

LOWER BOUNDS ON COLORING NUMBERS FROM HARDNESS HYPOTHESES IN PCF THEORY

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ABSTRACT. We prove that the statement “for every infinite cardinal ν , every graph with list chromatic ν has coloring number at most $\beth_\omega(\nu)$ ” proved by Kojman [6] using the RGCH theorem [11] implies the RGCH theorem via a short forcing argument. Similarly, a better upper bound than $\beth_\omega(\nu)$ in this statement implies stronger forms of the RGCH theorem hold, whose consistency and the consistency of their negations are wide open.

Thus, the optimality of Kojman’s upper bound is a purely cardinal arithmetic problem, and, as discussed below, is hard to decide.

1. INTRODUCTION

Recall that the *list-chromatic* or *choosability* number of a graph $G = \langle V, E \rangle$ is κ if κ is the least cardinal such that for any assignment of lists of colors $L(v)$ to all vertices $v \in V$ such that $|L(v)| \geq \kappa$ there exists a proper vertex coloring c of G with colors from the lists, namely $c(v) \in L(v)$ for all $v \in V$. A graph G has *coloring number* κ if κ is the least cardinal such that there exists a well-ordering $<$ on V such that a vertex $v \in V$ is joined by edges to only $< \kappa$ vertices u satisfying $u < v$.

Alon [1] proved that every finite graph with list-chromatic number n has coloring number at most $(4 + o(1))^n$ and this bound is tight up to a factor of $2 + o(1)$ by [3].

In [6] Kojman used the Revised GCH theorem from cardinal arithmetic [11] to prove in ZFC the upper bound of $\beth_\omega(\nu)$ on the coloring number of any graph with a list chromatic number $\leq \nu$, where $\beth_\omega(\nu)$ is the cardinal gotten by applying the exponent function to ν infinitely many times.¹

By Erdős and Hajnal [2] from 1966, if the GCH is assumed, $(2^\nu)^+ = (\beth_1(\nu))^+$ bounds the coloring number of every graph with list-chromatic number ν for every infinite ν . It is now known that much weaker axioms than the GCH — certain weak consequences of the Singular Cardinals Hypothesis — imply the same upper bound (see the second section in [6]), so in “many” models of set theory, the upper bound is $(2^\nu)^+$. Komjath [5] recently improved the GCH upper bound to $2^\nu = \nu^+$, constructed models of

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¹Formally, $\beth_0(\nu) = \nu$, $\beth_{n+1}(\nu) = 2^{\beth_n(\nu)}$ and $\beth_\omega(\nu) = \lim_n \beth_n(\nu)$.

the GCH in which $\chi_\ell(G) = \text{Col}(G)$ for every graph with infinite $\chi_\ell(G)$ and showed that in MA models 2^ν is required.

The gap between the single exponent occurring in the finite case and in the infinite case with additional mild cardinal arithmetic axioms, on the one hand, and the infinite tower of exponents in ZFC, on the other hand, led Kojman to ask whether the upper bound $\beth_\omega(\nu)$ could be lowered in ZFC and whether the use of the RGCH was necessary in proving this bound.

We prove here that: (1) the graph-theoretic upper bound implies the RGCH theorem; (2) a better upper bound implies a strengthening of the RGCH theorem. Both implications are via standard forcing arguments.

Thus, Kojman's upper bound on the coloring number and the question of its optimality are equivalent, respectively, to the cardinal arithmetic RGCH and the question of its own optimality. A better upper bound cannot be gotten, then, with only graph-theoretic arguments, and the cardinal arithmetic improvements which are necessary for improving the bound are hard. If all those problems are undecidable in ZFC, then Kojman's \beth_ω bound is optimal (see discussion below).

1.1. Description of the reduction. For a natural number m , consider the following two equivalent statements.

- (*)_m there is no cardinal ν and set \mathfrak{a} of $\beth_m(\nu)$ regular cardinals, each larger than $\beth_m(\nu)$, such that $J_{<\sup \mathfrak{a}}[\mathfrak{a}] \subseteq [\mathfrak{a}]^{<\nu}$, i.e. $\mathfrak{b} \in [\mathfrak{a}]^\nu$ implies that $\max \text{pcf}(\mathfrak{b}) \geq \sup \mathfrak{a}$.
- (**)_m there are no cardinals ν and μ satisfying $\beth_m(\nu) \leq \mu < \mu^+ < \beth_{m+1}(\nu)$ and a family of sets $\mathcal{A} \subseteq [\mu]^\mu$ such that $|A \cap B| < \nu$ for all distinct $A, B \in \mathcal{A}$ and $|\mathcal{A}| > \mu$.

Lowering $\beth_\omega(\nu)$ to $\beth_n(\nu)$ for some $n < \omega$ in the upper bound is at least as hard as proving that for $m \geq 2n + 1$, the statement (*)_m, or, equivalently, (**) _m, is not consistent. If the configuration forbidden by (**) _m exists in a model \mathbf{V} of ZFC then in some forcing extension of \mathbf{V} there is a graph with list-chromatic number θ and coloring number $> \beth_n(\theta)$, for some $\theta > \nu$. The relation $m \geq 2n + 1$ can probably be relaxed, but we made no effort to do so. The RGCH theorem implies, of course, that if m is replaced by ω in these statements they are no longer consistent. A similar forcing argument shows that if a “bad” configuration with ω instead of m exists in \mathbf{V} then in some forcing extension of \mathbf{V} there is a graph with list-chromatic number ν and coloring number $> \beth_\omega(\nu)$, so the RGCH follows from the graph-theoretic bound quite simply.

We discuss next the pcf-theoretic statements and explain their connection to upper bounds on coloring numbers.

Let $\kappa \leq \mu \leq \chi < \lambda = \text{cf}(\lambda)$ be cardinals. Consider the statement:

- (st) _{$\kappa, \mu, \chi, \lambda$} ¹ there is a $\mathcal{A} \subseteq [\chi]^\mu$ of cardinality λ such that if $A_1 \neq A_2$ belong to \mathcal{A} then $|A_1 \cap A_2| < \kappa$.

We agree that if $\lambda = \chi^+$ we may omit it and if $\mu = \chi$, $\lambda = \chi^+ = \mu^+$ then we also may omit them, so the typical case $(st)_{\kappa, \mu}^1$ is the existence of a family $\mathcal{A} \subseteq [\mu]^\mu$ of cardinality μ^+ which is a κ -family, that is, the intersection of any two distinct members of \mathcal{A} has cardinality $< \kappa$.

Why is using $(st)_{\kappa, \mu}^1$ reasonable when $\beth_m(\kappa) \leq \mu < \beth_{m+1}(\kappa)$? The history of this question is rich. In particular, Baumgartner got by forcing, without using large cardinals, the consistency of $(st)_{\kappa, \mu}^1$ with $\kappa = \kappa^{<\kappa} < \mu < 2^\kappa$, so here $m = 0$.

We are, however, interested in the cases $m \geq 1$, which are closely related to pcf problems.

Consider the pcf statemet,

- $(*)_{\kappa, \mu, \chi, \lambda}^2$ $\kappa < \mu < \chi < \lambda = \text{cf}(\lambda)$ and there is a sequence $\bar{\lambda} = \langle \lambda_i : i < \mu \rangle$ of regular cardinals with each $\mu < \lambda_i < \chi$ such that $\langle \prod_{i < \mu} \lambda_i, <_{[\mu]^{<\kappa}} \rangle$ has true cofinality λ (so really $\chi \gg \mu$. The main case, and the one we shall deal with, for transparency, is $\lambda = \chi^+$.)

Why $(st)_{\kappa, \mu, \chi, \lambda}^1$ and $(*)_{\kappa, \mu, \chi, \lambda}^2$ are related to each other and to graph colorings?

- $(*)_0$ if $\mathcal{A} \subseteq [\chi]^\mu$ has cardinality $> \chi$, and is a κ -family, $\kappa \leq \mu \leq \chi$ then the natural graph associated to \mathcal{A} and denoted $G_{\mathcal{A}}$, (see definition 2.4 below) has coloring number $\geq \chi^+$.

So finding such \mathcal{A} with small list-chromatic number, say ν , with $\beth_n(\nu) \leq \lambda = \chi^+$, will give consistent lower bounds, which is the purpose of this note. The main point here is that the list-chromatic number of such graphs can be lowered by applying the internal forcing axiom from [13], a natural generalization of MA.

Observe that

- $(*)_1$ If $(st)_{\kappa_1, \mu_1, \chi_2}^\ell$ and $\kappa_1 \leq \kappa_2 \leq \mu_2 \leq \mu_1$ then $(*)_{\kappa_2, \mu_2, \chi}^\ell$.
- $(*)_2$ (a) $(st)_{\kappa, \mu, \chi}^2$ implies $(st)_{\kappa, \mu, \chi}^1$.
- (b) If $(st)_{\kappa, \mu, \chi}^\ell$ and $\chi = \chi_1^+$, $\mu_1 = \min\{\mu, \chi_1\} \geq \kappa$ (so $\ell = 1 \Rightarrow \mu_1 = \mu$) then $(st)_{\kappa, \mu_1, \chi_1}^\ell$.
- (c) If $(st)_{\mu, \kappa, \chi, \lambda}^\ell$ and $\mu < \chi < \text{cf}(\lambda)$ and χ is a limit cardinal of cofinality $\neq \text{cf}(\mu)$ then for every large enough $\chi_1 < \kappa$ we have $(st)_{\kappa, \mu, \chi_1, \lambda}^\ell$.

Also

- $(*)_3$ If $2^\mu < \lambda = \text{cf}(\lambda)$ and $(st)_{\kappa, \mu, \chi, \lambda}^1$ then $(st)_{\kappa, \mu, \chi, \lambda}^2$.

See [10], 6.1.

Let

- $(*)_{\kappa, \mu}^{0, n}$ $\mu \in (\beth_n(\kappa), \beth_{n+1}(\kappa))$.

So the problem with the consistency of $(*)_{\kappa, \mu}^1 + (*)_{\kappa, \mu}^{0, n}$ is having $(*)_{\kappa, \mu, \chi}^2 + (*)_{\kappa, \mu}^{0, n}$.

An example, then, of how this note clarifies the question of whether the upper bound of $\beth_\omega(\nu)$ is tight is:

Conclusion 1.1. *We have $(A) \iff (B) \iff (C)$ where:*

- (A) *For every n in some forcing extension of \mathbf{V} there are $\kappa, \theta = \beth_n(\kappa)$, $\mu > \theta$ and a κ -family $\mathcal{A} \subseteq [\mu]^\theta$ of cardinality $> \mu$.*
- (B) *For every n in some forcing extension of \mathbf{V} there are $\kappa, \theta = \beth_n(\kappa)$ and a set \mathfrak{a} of θ regular cardinals $> \theta$ such that $J_{<\sup \mathfrak{a}}[\mathfrak{a}] \subseteq [\mathfrak{a}]^{<\kappa}$, i.e. $\mathfrak{b} \in [\mathfrak{a}]^\kappa$ implies that $\max \text{pcf}(\mathfrak{b}) \geq \sup \mathfrak{a}$.*
- (C) *For every n in some forcing extension of \mathbf{V} there are $\kappa, \theta = \beth_n(\kappa)$ and a graph G with list-chromatic number κ and coloring number $> \theta$.*

Proof of 1.1. $(A) \implies (B)$ follows from [10], 6.1 (and $(B) \implies (A)$ is obvious by $(*)_2$ above).

$(A) \implies (C)$ is done below.

To prove $(C) \implies (B)$ it suffices to note, (use $\theta = \theta^{<\theta}$) that $(a)_{\lambda, \theta, \kappa} \Rightarrow (b)_{\lambda, \theta, \kappa}$ in Claim 2.13. See [12]. A proof of compactness in singulars [9] and [14], Section 2. \square

In conclusion, the upper bound $\beth_\omega(\nu)$ cannot be lowered without making substantial progress in pcf theory. If, on the other hand, the negations of $(**)_{\mathfrak{m}}$ are consistent for all \mathfrak{m} , then Kojman's $\beth_\omega(\nu)$ upper bound is optimal.

1.2. Should we expect consistency or better pcf theorems? Let us mention first the known consistency results. Only quite recently Gitik [?] succeeded to prove, from the consistency of large cardinal axioms, the consistency of a countable set of regular cardinals \mathfrak{a} with $\text{pcf}(\mathfrak{a})$ uncountable, but really just $|\text{pcf}(\mathfrak{a})| = \aleph_1$. In particular he got $(*)_{\aleph_0, \aleph_1, \mu}^2$. While a great achievement, this is still very distant from what we need.

For $\kappa > \aleph_0$ there are no known consistency results. However, after the RGCH was proved in the early nineties much effort (by me, at least) was made to lower \beth_ω and failed, whereas in some other direction there were advances ([15, 16, 4]).

So do we expect consistency or ZFC results? Wishful thinking, or, if you prefer, the belief that “set theory behaves in an interesting way” suggests that truth should turn out to be somewhere in the middle, e.g. that the true ZFC bound is, say, $\beth_4(\nu)$ (or $\beth_{957}(\nu)$, for that matter). More seriously, the situation is wide open. Perhaps, as on the one hand the ZFC $\beth_\omega(\nu)$ gap has not changed for a long time now, while on the other hand there has been a recent breakthrough in consistency, there is some sense in viewing consistency as more likely.

2. PROOFS

Theorem 2.1. *Suppose that $\kappa < \theta = \theta^{<\kappa}$, $\mu > \beth_{2\ell+1}(\theta)$ and there a κ -family $\mathcal{A} \subseteq [\mu]^\mu$ of size μ^+ . Then in some forcing extension there is a graph G with list-chromatic number θ and coloring number $> \beth_\ell(\theta)$.*

Theorem 2.2. *Suppose that $\kappa < \theta = \theta^{<\kappa}$, $\mu > \beth_\omega(\theta)$ and there is a κ -family $\mathcal{A} \subseteq [\mu]^\mu$ of size μ^+ . Then in some forcing extension there is a graph G with list-chromatic number θ and coloring number $> \beth_\omega(\theta)$.*

Convention: For this section we fix $\aleph_0 \leq \kappa < \theta$.

We shall need the following definition from [13] p. 5. (See also [17] for more on this and other forcing axioms).

Definition 2.3. *A forcing notion \mathbb{P} satisfies $*_\mu^\omega$ for $\omega < \mu = \text{cf}(\mu)$ if Player I (the "completeness" player) has a winning strategy in the following game in ω moves:*

At step k : If $k \neq 0$ then Player I chooses $\langle p_{1,i}^k : i < \mu^+ \rangle$ with $p_{1,i}^k \in \mathbb{P}$ such that for all $\xi < \zeta$ and for club-many $i < \mu^+$ in $S_\mu^{\mu^+}$, $p_{2,i}^k \leq p_{1,i}^k$, and also chooses a function $f_k : \mu^+ \rightarrow \mu^+$ which is regressive on a club of μ^+ . If $k = 0$ Player one chooses $p_1^0 = \emptyset_{\mathbb{P}}$ and f^0 as the identically 0 function on μ^+ .

Player II chooses $\langle p_{2,i}^k : i < \mu^+ \rangle$ such that for club many $i < \mu^+$ in $S_\mu^{\mu^+}$ it holds that $p_{1,i}^k \leq p_{2,i}^k$.

Player I wins if there is a club $E \subseteq \mu^+$ such that for all $i < j$ in $E \cap S_\mu^{\mu^+}$, if $f_k < (i) = f_k(j)$ for all $k < \omega$ then there is an upper bound in \mathbb{P} to the set $\{p_{1,i}^k : k < \omega\} \cup \{p_{2,i}^k : k < \omega\}$.

Definition 2.4. (1) \mathcal{A} is a κ -family of sets when $|A \cap B| < \kappa$ for all distinct $A, B \in \mathcal{A}$ and is a (θ, κ) -family if in addition $|A| = \theta$ for all $A \in \mathcal{A}$.

(2) Suppose \mathcal{A} is a κ -family of sets and $\mathcal{A} \cap \bigcup \mathcal{A} = \emptyset$. The (bipartite) graph $G_{\mathcal{A}}$ has vertices $V_{\mathcal{A}} = \mathcal{A} \cup \bigcup \mathcal{A}$. We denote $\bigcup \mathcal{A}$ by $\text{pt}(\mathcal{A})$. The edge set $E_{\mathcal{A}}$ is $\{\{v, A\} : v \in A \in \mathcal{A}\}$. When \mathcal{A} is fixed or clear from context, we refer to $G_{\mathcal{A}}$ as $\langle V, E \rangle$.

Definition 2.5. For a (θ, κ) -family \mathcal{A} , a set $Y \subseteq G_{\mathcal{A}}$ is closed if:

- (1) $A \neq B \in Y \Rightarrow A \cap B \subseteq Y$.
- (2) If $|A \cap Y| \geq \kappa$ then $A \in Y$.

A subgraph G' of $G_{\mathcal{A}}$ is closed if its set of vertices is closed.

Claim 2.6. *If \mathcal{A} is a (θ, κ) -family and $\lambda^\kappa = \lambda \geq \theta$ then every subgraph of $G_{\mathcal{A}}$ of size λ is contained in a closed subgraph of the same size. Moreover, if $Y_1 \subseteq G$ is closed and $X \subseteq Y_1$ is of size λ , there there is a closed $Y \subseteq Y_1$ of cardinality λ such that $X \subseteq Y$.*

Remark: instead of $\lambda^\kappa = \lambda$ it suffices that $\mathcal{D}(\lambda, \kappa) = \lambda$, where $\mathcal{D}(\lambda, \kappa) = \text{cf}([\lambda]^\kappa, \supseteq)$, (see [6]).

Definition 2.7. *Suppose $\theta > \kappa$ and μ are cardinals and $|\alpha|^\theta < \mu$ for all $\alpha < \mu$. We say that $\text{Pr}_{\theta, \kappa}(\mu)$ holds if for every (θ, κ) -family \mathcal{A} and every [closed] $Y \subseteq G_{\mathcal{A}}$ of cardinality $|Y| < \mu$ the list chromatic number of Y is at most θ , that is, for every assignments of lists $L(v)$ to vertices in $G_{\mathcal{A}}$ such that $|L(v)| \geq \theta$ there is a valid coloring $c \in \prod_v L(v)$.*

Claim 2.8. *Assume that Y is closed, $\text{cf}(\delta) \neq \text{cf}(\kappa)$, $\delta < \text{cf}(\mu)$ and $Z_i \in [Y]^{<\mu}$ increasing with $i < \delta$. If each Z_i is \mathcal{A} -closed then $Z := \bigcup_{i < \delta} Z_i$ is \mathcal{A} -closed.*

Proof. First, if $A, B \in Z \cap \mathcal{A}$ then for some $i < \delta$ it holds that $A, B \in Z_i$, hence $Z \cap B \subseteq Z_i \subseteq Z$. Second, if $A \in \mathcal{A}$ satisfies that $|A \cap Z| \geq \kappa$ then for some $i < \delta$ it holds that $|A \cap Z_i| \geq \kappa$ and as Z_i is closed, $A \in Z_i \subseteq Z$. \square

Lemma 2.9 (Step-up Lemma). *Suppose that $\mu = \mu^{<\mu} > \theta > \kappa$ and $\omega < \mu$ is a limit ordinal. Assume that*

- (1) *The internal forcing axiom for posets that satisfy $*_\mu^\omega$ from [13] holds for $< \lambda$ dense sets.*
- (2) $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$.
- (3) $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$
- (4) $\text{Pr}_{\theta, \kappa}(\mu)$ holds.

Then $\text{Pr}_{\theta, \kappa}(\lambda)$.

Proof. Suppose \mathcal{A} is as above and $Y \subseteq G_{\mathcal{A}}$ is closed, $|Y| < \lambda$ and $L(v)$ such that $|L(v)| = \theta$ is given for all $v \in Y$. We need to prove the existence of a valid coloring c of G such that $c(v) \in L(v)$ for all $v \in Y$.

Let \mathbb{P} be the following poset. $q \in \mathbb{P}$ iff q is a partial valid coloring from the given lists and $\text{dom}(q) \subseteq G$ is closed of cardinality $< \mu$. A condition q is stronger than a condition p , $q \geq p$, iff $p \subseteq q$.

(*)₂ \mathbb{P} is a forcing notion.

(*)₃ (Density) if $p \in P$ and $Z \subseteq Y$ satisfies $|Z| < \mu$ then there is $q \geq p$ such that $Z \subseteq \text{dom}(q)$.

Proof of (*)₃: By increasing Z , we may assume that Z is closed in Y and that $\text{dom}(p) \subseteq Z$. As $\text{dom}(p)$ is closed, for all $A \in \mathcal{A} \cap Z \setminus \text{dom}(p)$ it holds that $|A \cap \text{dom}(p)| < \kappa$. For $A \in Z \setminus Y$ let $L'(A) = L(A) \setminus \{p(v) : v \in A \cap \text{dom}(p)\}$. As $|A \cap \text{dom}(p)| < \kappa < \theta$ it holds that $|L'(A)| = \theta$.

For $v \in (Z \setminus Y) \cap \text{pt}(\mathcal{A})$ there is at most one $A \in \text{dom}(p)$ such that $v \in A$. Let $L'(v)$ be gotten from $L(v)$ by subtracting $\{c(A)\}$ from $L(v)$ for that unique A , when A exists. For all $x \in \text{dom}(p)$ let $L'(x) = L(x)$.

By $\text{Pr}_{\theta, \kappa}(\mu)$ there is condition p'' with $\text{dom}(p'') = Z$ such that $p''(v) \in L'(v)$. Let $p' = p'' \restriction (Z \setminus \text{dom}(p))$ and let $q = p \cup p'$. Now we claim that $q \in P$. As $q(x) \in L'(x) \subseteq L(x)$ for all $x \in \text{dom}(q)$, all that needs to be checked is the validity of the coloring q . Suppose that $v \in A$ and $v, A \in \text{dom}(q)$. First assume that $v \in \text{dom}(p)$ and $A \in \text{dom}(p')$. In this case $p(v) \neq p'(A)$ because $p(v) \notin L'(A)$ by the definition of $L'(A)$. Another case to check is $v \in \text{dom}(p')$ and $A \in \text{dom}(p)$, which followed from the choice of $L'(v)$. The two remaining cases are clear.

(*)₄ If $\langle p_i : i < \delta \rangle$ is an increasing sequence of conditions in \mathbb{P} and $\text{cf}(\delta) \neq \text{cf}(\kappa)$ then the union is a condition.

Let $Y_\delta = \bigcup \{\text{dom}(p_i) : i < \delta\}$. Now $|Y_\delta| < \mu$ as $i < \mu$ by the assumptions, and $i < \delta \Rightarrow \text{dom}(p_i) \in [G_{\mathcal{A}}]^{<\mu}$, recalling that μ is regular (see clause (1) of the claim's assumptions). Since $\text{cf}(\delta) \neq \kappa$, it holds that $p = \bigcup_i p_i$ is a condition.

(*)₅ If $\delta < \mu$, $\bar{p} = \langle p_i : i < \delta \rangle$ is increasing in \mathbb{P} and $\text{cf}(\delta) = \text{cf}(\kappa)$ then \bar{p} has an upper bound in \mathbb{P} .

Let $Z \subseteq G_{\mathcal{A}}$ be closed such that $|Z| < \mu$ and $Y_\delta = \bigcup_{i < \delta} \text{dom}(p_i) \subseteq Z$. By restricting to a subsequence we assume that $\delta = \kappa$ and so $A \in \mathcal{A} \setminus Y \Rightarrow \bigwedge_{i < \kappa} (|A \cap \text{dom}(p_i)| < \kappa \Rightarrow |A \cap Y| \leq \kappa)$. Now repeat the proof of (*)₃ with $p = \bigcup_i p_i$ with the following changes:

- (a) if $A \in Z \setminus Y$, $A \in \mathcal{A}$, then $|A \cap Y| \leq \kappa$ hence $L'(A) = L(A) \setminus \{p(v) : v \in A \cap Y\}$ has cardinality θ as $L(A)$ has cardinality $\theta > \kappa \geq |A \cap Y| \geq |\{p(v) : v \in A \cap Y\}|$.
- (b) if $v \in Z \setminus Y$, $v \in pt(G_{\mathcal{A}})$, then

$$i < \kappa \Rightarrow |\{A \in \text{dom}(p_i) \cap \mathcal{A} : v \in A\}| \leq 1,$$

hence

$$|\{A \in \mathcal{A} \cap Y : v \in A\}| \leq 1$$

and $L'(v) = L(v) \setminus \{p(A) : A \in Y_\delta \wedge v \in A\}$ has cardinality θ .

Now we can conclude as in (*)₃.

(*)₆ $\{p_\zeta^\ell : \ell = 1, 2 \text{ and } \zeta < \delta\}$ has a common upper bound when

- (a) $\delta < \kappa^+ \leq \mu$ (we will use $\delta = \omega < \kappa^+$ when simpler).
- (b) $p_\zeta^\ell \in \mathbb{P}$
- (c) $\zeta < \xi < \delta \Rightarrow p_\zeta^\ell \leq_{\mathbb{P}} p_\xi^\ell$.
- (d) p_ζ^1, p_ζ^2 are compatible functions for $\zeta < \delta$.

Let $p = \bigcup_{\ell, \zeta} p_\zeta^\ell$, so p is a function, but not necessarily a condition in \mathbb{P} . Let $Y = \text{dom}(p)$ and $Z \supseteq Y$ be closed and of cardinality $< \mu$.

(A) If $A \in Z \setminus Y$, $A \in \mathcal{A}$, then $\ell \in \{1, 2\} \wedge \zeta < \delta \Rightarrow |A \cap \text{dom}(p_i^\ell)| < \kappa$ so $\langle |A \cap \text{dom}(p_i^\ell)| : i < \delta \rangle$ is a non-decreasing sequence of conditions $< \kappa$ hence $\leq \kappa$. $|A \cup \bigcup_i \text{dom}(p_i^\ell)| \leq \kappa$, hence $|A \cap Y| \leq \kappa$.

(B) If $v \in Z \setminus Y$, $v \in pt(G_{\mathcal{A}})$, then $|\{A \in \bigcup_i \text{dom}(p_i^\ell), v \in A\}| \leq 1$ hence $|\{A \in Z \setminus A : A \in \mathcal{A}, v \in A\}| \leq 2$. So all is fine.

We continue as in the proof of (*)₅.

(*)₇ \mathbb{P} is μ -complete (by (*)₄ + (*)₅).

(*)₈ The property $*_\mu^\omega$ holds for \mathbb{P} .

The game which defines $*_\mu^\omega$ lasts ω steps and at each step $k < \omega$ we have a sequence of conditions $\langle p_{1,i}^\kappa : i < \mu^+ \rangle$, a club $E_\zeta \subseteq \mu^+$ and a regressive function $f_\zeta : (E \cap S_\omega^{\mu^+})$ played by the completeness player I (see [13] p. 5. See also [17] for more on this and other forcing axioms).

This is how player I chooses E_k and f_k . E_k is sufficiently closed.

$f_k : E \cap S_\mu^{\mu^+}$ is regressive such that:

⊕ If $\alpha_1, \alpha_2 \in \text{dom}(f_k)$, $f_k(\alpha_1) = f_\zeta(\alpha_2)$ then $p_{\alpha_1}^k, p_{\alpha_2}^k$ are compatible functions.

This clearly suffices (as the $\langle (p_{\alpha_1}^k, p_{\alpha_2}^k) : k < \delta \rangle$ are like $\langle (p_\zeta^1, p_\zeta^2) \rangle$ in (*)₆).

Clearly such a function exists.

$f_\zeta(\delta)$ codes:

- (1) $a_\delta^k = \text{dom}(p_\delta^k) \cap (\bigcup_{\alpha < \delta} \text{dom } p_\alpha^k)$.
- (2) $p_\delta^\zeta \upharpoonright a_\delta^\zeta$.
- (3) $\text{ran}(h_\delta^\zeta) \cap \text{pt}(\mathcal{A}) \setminus a_\delta^\zeta$ where $\text{dom } h_\delta^k = \text{dom}(p_\delta^k) \cap \text{pt}(\mathcal{A}) \setminus a_\delta^k$ and $h_\delta^k(t) = \min\{\alpha < \delta : (\exists A)(A \in \mathcal{A} \cap \text{dom}(p_\alpha^k \wedge t \in A))\}$.

So now player I wins as whenever $i < j$ belong to $S_\omega^{\mu^+} \cap \bigcap_k E_k$ and $\bigwedge_k f_k(i) = f_k(j)$, the set of conditions $\{p_{1,i}^k : k < \omega\} \cup \{p_{2,i}^k : k < \omega\}$ has an upper bound in P .

This proves $(*)_5$.

By the axiom for posets with $*_\varepsilon^\mu$, there is a generic filter for \mathbb{P} which meets all dense sets D_x for $x \in Y$, where $p \in D_x$ if $x \in \text{dom}(p)$. The union of the generic is a valid coloring from the lists on Y . \square

Corollary 2.10. *Suppose $n \geq 1$ and*

- (1) $\mu_0 < \mu_1 \cdots < \mu_n$.
- (2) *For all $l \leq n-1$ it holds that $(\forall \alpha < \mu_l)(|\alpha|^\kappa < \mu_l)$.*
- (3) $2^{\mu_i} = \mu_{i+1}$ for $i < n$.
- (4) *For every $i < n$, the forcing axiom for posets with $*_{\mu_i}^\varepsilon$ and $< \mu_{i+1}$ dense sets holds.*
- (5) $\mu_0 \leq \theta^+$.

Then $\text{Pr}_{\theta, \kappa}(\mu_n)$.

Proof. By induction on n . Since the list-chromatic number of any graph G of cardinality $< \mu_0$ is $\leq |G| \leq \theta$, the condition $\text{Pr}(\mu_0)$ holds trivially. The induction step follows from the main lemma. \square

Next we show how to force the conditions of the previous lemma.

Claim 2.11. *Assume that:*

- (a) $\theta = \theta^{<\kappa} > \kappa$ and $\beth_{2n+1}(\theta) < \mu \leq \chi < \lambda$.
- (b) $(st)_{\kappa, \mu, \chi, \lambda}^1$.

Then: For some \mathbb{P}

- (a) \mathbb{P} is a θ^+ -complete forcing notion that satisfies $(\beth_{2n+1}(\theta))^+$ -c.c.
- (b) in $\mathbf{V}^\mathbb{P}$ we have $\mu_\ell = (\beth_\ell(\theta))^+ < \beth_{\ell+1}(\theta)$ for all $\ell \leq n$, $|\alpha|^\kappa < u_\ell$ for all $\alpha < u_\ell$, and $\beth_n(\theta) < \mu \leq \chi < \beth_{n+1}(\theta)$.
- (c) $(st)_{\kappa, \mu, \chi, \lambda}^1$.
- (d) The forcing axiom $*_{\mu_\ell}^\omega$ with $< \mu_{\ell+1}$ holds for all $\ell \leq n$.

Proof. Clearly,

$$(*)_1 \quad (\beth_{2n+1}(\theta))^+ < \mu.$$

Now let

- (*)₂ (a) $\mu_\ell = (\beth_{2\ell}(\theta))^+$ for $\ell \leq n$ so $2^{<\mu_\ell} \leq \beth_{2\ell+1}(\theta)$.
- (b) Choose $\mu_{n+1} = \text{cf}(\mu_{n+1}) = (\mu_{n+1})^{(\mu_n^\kappa)} > \lambda$ such that $\alpha < \mu_{n+1} \Rightarrow |\alpha|^\kappa < \mu_{n+1}$

Remark: less suffices. $\mu_{n+1} = (\lambda^\kappa)^+$ or just $\mu_{n+1} = \text{cf}(\mu_{n+1}) > \lambda$ satisfies $(\forall \alpha < \mu_{n+1})(|\alpha|^\kappa < \mu_{n+1})$, but will complicate the notation below, e.g. $(*)_4(b)$ for $\ell = n$ is different.

Now

- (*)₃ (a) $\mu_0 = \theta^+$ hence $\mu_0 = \text{cf}(\mu_0)$ and $(\forall \alpha)(\alpha < \mu_0 \rightarrow |\alpha|^\kappa \leq \theta^\kappa < \mu_0)$.
- (b) $\mu_0 < \mu_1 < \dots < \mu_n < \mu_{n+1}$ are regular.
- (c) $(\forall \alpha < \mu_\ell)(|\alpha|^\kappa < \mu_\ell)$ for all $\ell \leq n+1$.
- (d) $(\mu_{\ell+1})^{2^{<\mu_\ell}} = \mu_{\ell+1}$.
- (e) $\mu_n < \mu \leq \chi < \lambda < \mu_{n+1}$.

Let

- (*)₄ Let (a) $\mathbb{Q}_\ell^* = \text{Levy}(\mu_\ell, 2^{<\mu_\ell})$ for $\ell \leq n$.

- (b) $\mathbb{Q}_* = \prod_{\ell \leq n} \mathbb{Q}_\ell$.

Easily,

- (*)₅ (a) \mathbb{Q}_ℓ^* is μ_ℓ -complete and of cardinality 2^{μ_ℓ} .

- (b) In $\mathbf{V}_\ell := \mathbf{V}^{\prod_{k < \ell} \mathbb{Q}_k^*}$

We work from now on in $\mathbf{V}_{n+1} := \mathbf{V}^{\prod_{\ell \leq n} \mathbb{Q}_\ell}$.

- (*)₆ We define $\langle (\mathbb{P}_k, \mathbb{Q}_\ell, \mathbb{Q}_\ell^2) : k \leq n+1, \ell \leq n \rangle$ such that :

- (*)₇] (a) \mathbb{P}_0 is the trivial forcing.

- (b) $\mathbb{P}_{\ell+1}$ is a forcing notion of cardinality μ_{n+1} .

- (c) $\mathbb{P}_{\ell+1}$ satisfies the μ_ℓ^+ -c.c.

- (d) $\mathbb{P}_{\ell+1} = \mathbb{P}_\ell * \mathbb{Q}_\ell^2$.

- (e) \mathbb{Q}_ℓ^2 is a $\mathbb{P}_{<\ell}$ -name of a forcing notion of cardinality $\mu_{\ell+1}$ that satisfies

μ_ℓ^+ -c.c. that forces $2^{\mu_\ell} = \mu_{\ell+1}$ and the axiom for forcing notions that satisfy $*_{\mu_\ell}^\omega$ for $< \min\{\mu_{\ell+1}, (\mu^\kappa)^+\}$ dense sets.

There is no problem to carry the induction (note that $(\mu_{\ell+1})^{<\mu} = \mu_{\ell+1}$ in $\mathbf{V}_{n+1}^{\mathbb{P}_{n+1}}$.) We return to \mathbf{V} . In \mathbf{V} we have a \mathbb{Q}_k -name \mathbb{P}_{n+1} for \mathbb{P}_{n+1} . Let, in \mathbf{V} , $\mathbb{P} = \mathbb{Q}_* * \mathbb{P}_{n+1}$. Why \mathbb{P} is as required?

Clearly, all forcing notions $\mathbb{Q}_\ell^*, \mathbb{Q}_*, \mathbb{P}_{n+1}, \mathbb{P}$ are θ^+ -complete, hence so is $\mathbf{V}^\mathbb{P}$. Therefore, $(\forall \alpha < \mu_\ell)(|\alpha|^\kappa < \mu_{\ell+1})$ for all $\ell < n+1$ because we prove below that μ_ℓ does not collapse.

Clearly, \mathbb{P} has cardinality μ_{n+1} and $\Vdash_{\mathbb{P}} \mu_\ell = \mu_\ell^{<\mu_\ell}$ is not collapsed, and \mathbb{P} satisfies the $((2^{<\mu_n})^+)$ -c.c. as \mathbb{Q}_* does, and \mathbb{P}_{n+1} satisfies μ^+ -c.c.

Lastly, the relevant forcing axiom holds: if $\ell < n$, the one for $(*)_{\mu_\ell}^\varepsilon$ and $< \mu_{\ell+1}$ -dense sets. So replacing μ_{n+1} by $(\mu^\kappa)^+$ and applying 2.6 we are done. \square

A similar argument works to replace n with ω :

Theorem 2.12. *The condition $(A)_{\ell(*)}$ implies the condition $(B)_{\ell(*)}$ for $\ell(*) \in \{1, 2\}$, where:*

- (A)₁ $\aleph_0 < \text{cf}(\kappa) \leq \kappa < \theta = \theta^{<\kappa}$, $\chi \geq \lambda \geq \beth_\omega(\kappa)$ and there exists a κ -family $\mathcal{A} \subseteq [\chi]^\lambda$ of cardinality $|\mathcal{A}| \geq \chi^+$.

- (A)₂ $\aleph_0 < \text{cf}(\kappa) \leq \kappa < \theta = \theta^{<\kappa}$ and for every $n < \omega$ there are $\chi_n > \lambda_n \geq \beth_n(\theta)$ a κ -family $\mathcal{A}_n \subseteq [\chi_n]^{\lambda_n}$ of cardinality $|\mathcal{A}_n| \geq \chi_n^+$.
- (B)₁ For some forcing notion \mathbb{P} not adding new sequences of ordinals of length $< \theta$, it holds that:
- $(\beth_\omega(\theta))^{\mathbf{V}^\mathbb{P}} = (\beth_\omega(\theta))^\mathbf{V}$.
 - There exists a graph G with list-chromatic number θ and coloring number $> (\beth_\omega(\theta))^+$.
- (B)₂ Like (B)₁ with the coloring number $\geq (\beth_\omega(\theta))^+$.

Proof. Stage A. For $(A)_1 \Rightarrow (B)_1$ assume $(A)_1$ and let $(\chi_n, \lambda_n) = (\chi, \lambda)$, $\mathcal{A}_n = \mathcal{A}$, so we can assume $(A)_2$.

- (*)₂ Let $u_1 = \{n : \lambda_n < \beth_\omega(\theta)\}$, $u_2 = \{n : \lambda_n = \beth_\omega(\theta)\}$ and $u_3 = \{n : \lambda_n > \beth_\omega(\theta)\}$.
- (*)₃ Without loss of generality, for some $\mathbf{i} \in \{1, 2, 3\}$ we have:
- (a) $u_{\mathbf{i}} = \omega$.
 - (b) If $\mathbf{i} = 3$ without loss of generality there is some $\lambda_* > \beth_\omega(\theta)$ such that $\bigwedge_n \lambda_n = \lambda_*$.
 - (c) If $\mathbf{i} = 2$ let $\mu_* = \beth_\omega(\theta)$.

Stage B. Now

- (*)₄ Without loss of generality there is a sequence $\langle \mu_n : n < \omega \rangle$ such that
- (a) $\mu_0 = \theta^+$.
 - (b) $\mu_n = \text{cf}(\mu_n)$.
 - (c) $2^{\mu_n} = \mu_{n+1}$.
 - (d) Hence $\sum_n \mu_n = \beth_\omega(\theta)$.
 - (e) The forcing axiom $*_{\mu_n}^\omega$ and $< \mu_{n+1}$ dense sets holds.

Why? As in the proof of 2.11.

- (*)₅ Without loss of genrality, in addition, letting $\theta_\omega = (\beth_\omega(\theta))$, we have $2^{\theta_\omega} = \theta_\omega^+$ and $\mu_{\omega+1} = 2^{\mu_\omega}$ is $> \sum_n \chi_n$ and as in (*)₄(e) the forcing axiom $*_{\theta_\omega^+}^\omega$ and $< \mu_{\omega+1}$ dense sets holds.

Stage C. We deal with the case $\mathbf{i} = 1$.

By 2.10, for every n , $Pr_{\theta, \kappa}(\mu_n)$ holds. By easy compactness for singulars argument we have, as $\aleph_0 < \text{cf}(\theta_*)$, also $Pr_{\theta, \kappa}(\mu_\omega)$. By 2.9 we have $Pr_{\theta, \kappa}(\mu_{\omega+1})$.

Now clearly for each n , $\chi_n < \mu_n$, as in the proof of Theorem 1, there is a graph G_n with $|\mathcal{A}_n|$ vertices, coloring number $\geq \lambda_n$ and list-chromatic number θ .

Taking then the disjoint sum of all G_n we have established $(A)_2 \Rightarrow (B)_2$.

Stage D. $\mathbf{i} \in \{2, 3\}$. Similarly, but we use (*)₅.

□

Remark: We can replace $\beth_\omega(\theta)$ with $\beth_{\delta(*)}(\theta)$ when $\delta(*) < \text{cf}(\kappa)$.

Proof of Theorems 1 and 2. The proofs consists of combining the lemmas above. □

We conclude with a few simple implications that are needed above.

Claim 2.13. *Assume that θ is a regular cardinal and $2^\kappa \leq \theta \leq \lambda$. We have $(a)_{\lambda,\theta,\kappa} \Rightarrow (b)_{\lambda,\theta,\kappa} \Rightarrow (c)_{\lambda,\theta,\kappa} \Rightarrow (d)_{\lambda,\theta,\kappa}$. If, in addition, $\theta = \theta^\kappa$ (or just $\mu < \theta \Rightarrow \mu^\kappa < \theta$ and $\partial < \theta \Rightarrow \mu^\partial < \lambda$) then $(d)_{\lambda,\theta,\kappa} \Rightarrow (e)_{\lambda,\theta,\kappa} \Rightarrow (f)_{\lambda,\theta,\kappa}$,
Where*

- (a) $_{\lambda,\theta,\kappa}$ λ is minimal such that there is a graph G with λ vertices, coloring number $> \theta$ and list-chromatic number $\leq \kappa$.
- (b) $_{\lambda,\theta,\kappa}$ λ is regular and there is a graph G with λ vertices, coloring number $\geq \theta$, every subgraph of G with $< \lambda$ vertices has coloring number $\leq \theta$ and the complete bipartite graph $K(\kappa, 2^\kappa)$ is not weakly embeddable into G .
- (c) $_{\lambda,\theta,\kappa}$ $\lambda > \theta$ is regular and there is \overline{C} such that:
 - (α) $\overline{C} = \langle c_\delta : \delta \in S \rangle$
 - (β) $S \subseteq \{\delta : \delta < \lambda \wedge \text{cf}(\delta) = \theta\}$ is stationary.
 - (γ) $C_\delta \subseteq \delta$ and $\text{otp}(C_\delta) = \theta$.
 - (δ) If $u \in [\lambda]^\kappa$ then $\{\delta \in S : u \subseteq C_\delta\}$ is bounded in λ .
- (d) $_{\lambda,\theta,\kappa}$ $\lambda > \theta$ is regular and for some $\mu < \lambda$ for every $\delta \in [\kappa, \theta)$ there is $\mathcal{A} \subseteq [\mu]^\delta$ of cardinality λ such that $u \in [\mu]^\kappa \Rightarrow (\exists^{<\lambda} v \in \mathcal{A})(v \subseteq u)$.
- (e) $_{\lambda,\theta,\kappa}$ $\lambda > \theta$ is regular and there are $\mu < \lambda$ and $\{A_\partial : \partial \in [\kappa, \theta)\}$ such that $A_\partial \subseteq [\mu]^\delta$ is a κ -family of cardinality λ .
- (f) $_{\lambda,\theta,\kappa}$ $\lambda > \theta$ is regular and there are $\mu < \lambda$ and $\{\mathfrak{a}_\delta : \delta \in [\kappa, \theta)\}$ such that $\mathfrak{a} \subseteq \text{Reg} \cap (\mu \setminus \theta)$, $|\mathfrak{a}_\alpha| = \delta$ and $(\prod \mathfrak{a}_\delta, <_{[\mathfrak{a}_\delta] < \kappa})$ is λ -directed.

Proof. (a) $_{\lambda,\theta,\kappa} \Rightarrow$ (b) $_{\lambda,\theta,\kappa}$. Choose G witnessing (a) $_{\lambda,\theta,\kappa}$. We know that λ is regular, and without loss of generality the vertex set of the graph is λ . The coloring number is $\geq \theta$ by the choice of G . If $H \subseteq G$ has fewer than λ vertices then it has coloring number $< \theta$ by the minimality of λ . Also the complete bipartite graph $K(\kappa, 2^\kappa)$ is not weakly embedded in G because its list-chromatic number is κ^+ and $\lambda > 2^\kappa$. Minimality of λ gives more. So (b) $_{\lambda,\theta,\kappa}$ holds.

(b) $_{\lambda,\theta,\kappa} \Rightarrow$ (c) $_{\lambda,\theta,\kappa}$. See [8] or [9]. Assume that the vertex set is λ and let $S = \{\delta : (\exists \alpha \geq \delta)(|G[\alpha] \cap \delta| \geq \theta)\}$. If S is not stationary then using "every subgraph with $< \lambda$ vertices has coloring number $\leq \theta$ " we conclude that G has coloring number $\leq \theta$. By renaming we get (c) $_{\lambda,\theta,\kappa}$.

(c) $_{\lambda,\theta,\kappa} \Rightarrow$ (d) $_{\lambda,\theta,\kappa}$. For each $\partial \in [\kappa, \theta)$ we find, by Fodor's lemma, $\alpha_\partial < \mu$ such that $\mathcal{A}_\gamma = \{\delta \in S : |C_\partial \cap \alpha_\partial| \geq \partial\}$ has cardinality λ . So $\alpha_* = \bigcup_\partial \alpha_\partial < \lambda$ satisfies the desired conclusion for $\mu = |\alpha_*|$ so by renaming we are done.

(d) $_{\lambda,\theta,\kappa} \Rightarrow$ (e) $_{\lambda,\theta,\kappa}$. When, e.g., $\partial < \theta \Rightarrow \partial^\kappa < \theta$ for each $\partial \in [\kappa, \theta)$ let $\langle u_{\partial,\alpha} : \alpha < \lambda \rangle$ list \mathcal{A}_γ , and for $\alpha < \lambda$ let $W_\alpha = \{\beta < \lambda : |u_{\gamma,\beta} \cap u_{\partial,\alpha}| \geq \kappa\}$. As $|u_{\partial,\alpha}|^\theta < \lambda = \text{cf}(\lambda)$, the set W_∂ is bounded in λ , hence for some club $E_\partial \subseteq \lambda$ it holds that $\alpha < \beta \in E_\partial \Rightarrow |u_{\gamma,\alpha} \cap u_{\partial,\beta}| < \kappa$, so $\{u_{\partial,\alpha} : \alpha \in E_\partial\}$ is as required.

(e) $_{\lambda,\theta,\kappa} \Rightarrow$ (f) $_{\lambda,\theta,\kappa}$ if $\partial < \theta \Rightarrow \partial^\kappa < \lambda$. By [10] 6.1.

□

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